Tracing Primary 6 Students' Model Development within the Mathematical Modelling Process

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Abstract
This study focused on how three groups of Primary 6 students manage a mathematical modelling task situated in a problem-based setting. This perspective saw mathematical modelling as a problem-solving activity. A macro level analysis of the students' modelling endeavour was carried out to construct and trace the students' model development with respect to their conceptual representations and mathematical relations. The results suggest that the models the students had developed were based on recognizing the structure between the quantities and variables in relation to the context. The models were tested and revised towards better or newer models with the aim of attaining the best solution model. The modelling process also revealed the students' mathematical and personal knowledge employed to facilitate their solution.

Keywords: mathematical modelling; problem-based learning; model development

1. Introduction

Mathematics education reform efforts worldwide are increasingly calling for pedagogies that teach for understanding where students' cognitive reasoning could be deeply engaged through problem-based learning (Erickson, 1999; Hiebert et al, 1996), modelling activities (English, 2006; Lesh & Zawojewski, 2007), open-ended problems (Chan 2007; Becker & Shimada, 1997), and teaching through problem solving (Lambdin, 2003). The rationale for change apparently is tied to equipping students for a knowledge-based workforce possessing competencies and skills beyond school to solve real-world problems (Chan, 2008a; English & Sriraman, 2009). Mathematical modelling progressively is becoming one of the focus areas in mathematics education (Barbosa, 2009) with some researchers advocating it to be the new direction for research in mathematical problem solving (Lesh & Zawojewski, 2007) and as well be regarded as among the most significant goals of mathematics and science education (Lesh & Sriraman, 2005; Niss, Blum, & Galbraith, 2007). Research in mathematical modelling takes on various perspectives (pragmatic, scientific-humanistic, emancipatory, integrative, socio-cultural, models-and-modelling, etc.) depending on research goals (Barbosa, 2009; Kaiser & Sriraman, 2006). However, research is lacking in terms of examining students' modelling endeavours (Barbosa, 2009; Lehrer & Schauble, 2003). Current research on younger learners, though limited, has provided a glimpse that children were able to mathematize and develop systems for sorting, quantifying, weighting, ranking and organizing (Chan, 2008b; English, 2006; English & Watters, 2005; Lehrer & Schauble, 2000). Modelling activities were also found to have helped in the promotion of important mathematical reasoning processes such as constructing, explaining, justifying, predicting, conjecturing, and representing (English & Watters, 2005); reasoning aspects that are valued as a powerful way to accomplishing learning with understanding. Furthermore, studies have shown that children have been able to manage complex mathematical constructs irrespective of their academic mathematics achievement (Lesh & Doerr, 2003). If mathematical modelling is to be seen as a way of making sense of the physical and social world, then there is a need to understand more through promoting ways for children to externalize their thinking of giving meanings to situations.

This paper is part of a larger study in investigating Primary 6 (Grade 6) students' mathematical modelling process and reports the model development aspect of three groups of Primary 6 students where modelling is seen as a problem-solving activity to support the mathematics curriculum.
2. Mathematical Modelling as a Problem-Solving Activity

Mathematical problem solving has been the core of many a mathematics curriculum since the 1980s and it still is. However, mathematics classroom practices that are dominantly characterized by the solving of structured word problems limit the notion of what problem-solving should encompass (Chan, 2009) as they tend to lend themselves to one way of interpreting the problem which do not address adequately the mathematical knowledge, processes, representational fluency, and social skills needed for the 21st century (English, 2003). A key aspect of reform-oriented mathematics programmes should comprise purposeful problem solving that is carried out within meaningful contexts towards sense-making and mathematical reasoning of data (Burrill & Romberg, 1998; Doerr & English, 2003).

This paper adopts a modelling perspective that sees mathematical modelling as a problem-solving activity. In this light, a problem-based instructional approach was used to situate the modelling activity. The problem-based approach was intentionally factored to highlight the significance of the task driving the learning (Tan, 2003; Stephien & Pyke, 1997; Hmelo-Silver, 2004). The task (a modelling task in this case) is differentiated from traditional structured word problems in that it is based on authentic and contextually rich information used as the stimulus for triggering deep and high extents of mathematical and metacognitive thinking. The task has to be engaged collaboratively as it appeals to different ways of managing it and can result in a variety of solutions. In a problem-based setting, the teacher facilitates rather than gives direct instructions.

Although research in the problem-based learning setting that involves younger participants is limited (Savery, 2006), researchers believe that the interaction of its key tenets, that is, the problem-based task, the student collaboration and the teacher facilitation promotes a learning milieu that encourages open-minded, reflective, critical and active learning (Margetson, 1991; Tan, 2003, Chan, 2008a). The strength of problem-based learning draws from situated cognition theory in that knowledge building is linked to the conditions present in the environment and the knowledge that one brings is part of an interacting system that is inclusive of individuals, relationships, tasks, tools, goals, teaching methods, and other contributing factors. Knowledge thus is seen as situated being in part a product of the activity, context, and culture in which it is developed and used (Brown, Collins, & Duguid, 1989).

The problem-based approach is also considered a compatible platform for mathematical modelling as the characteristics of the modelling tasks befit what problem-based tasks should entail (Hjalmanson & Diefes-Dux, 2008). Notably, mathematical modelling as problem solving involving young learners is seen in the works of Mousoulides, Sriraman and Christou (2007), Lesh & Doerr (2003), English & Doerr, (2003) and Chan (2009).

3. The Modelling Process

Figure 1 shows a generic mathematical modelling process. It is a broad and simplistic view of the process that is cyclical and involves four modelling stages, namely, Describing, Manipulating, Predicting and Verifying (Ang, 1991; Blum & Niss, 1991; Chan, 2008a; Lesh & Doerr, 2003; Mousoulides, Sriraman, Pittalis, & Christou, 2007; Swetz & Hartzler, 1991) although the terms used may be slightly different amongst some researchers. The common premise is that the starting point is a real world problem or situation that can be formulated into a mathematical problem and where the mathematical solutions are used to interpret the real-world situation.

![Figure 1. Modelling stages](image-url)
This modelling process however does not make obvious the various models that students can produce as emerging models which are revised and refined until an ideal solution model or product is attained. One view of mathematical modelling stemming from a models-and-modelling perspective (Lesh & Zawojewski, 2007) sees the students’ modelling process as going through multiple cycles in developing a mathematical model for a given problematic situation. The cycles of model construction, evaluation, and revision are valued in the light of befitting the professional practices of mathematicians and engineers (Lesh & Doerr, 2003) which supposedly convey a more realistic process of problem solving.

Models

A modelling perspective to mathematical problem solving focuses on the students’ representational fluency through the flexible use of mathematical ideas where the students have to make mathematical descriptions of the problem context and data. When students paraphrase, explain, draw diagrams, categorize, find relationships, dimensionalize, quantify, or make predictions, they are generally developing their conceptual systems or models through the mathematizing. As they work with the rich contextual data, they would need to surface and communicate their mathematical ideas to clarify their thoughts and weigh the validity of their ideas. In this sense, their “(internal) conceptual systems are continually being projected into the (external) world” (Lesh & Doerr, 2003, p. 11) thus making visible their sense-making systems of mathematical reasoning in the form of a variety of representational media such as spoken language, written symbols, graphs, diagrams, and experience-based metaphors. From this perspective, the modelling process results in the development of models in varying representations and where emerging models can be utilized to become part of or evolve into another model.

4. The Conceptual Framework

This paper is part of a larger study in investigating primary 6 students’ mathematical modelling process. This study adopts a conceptual framework that integrates the modelling approach into a problem-based learning (PBL) instructional setting. The PBL setting is to highlight the importance of the interaction between three essential tenets characteristic of problem-base learning; the modelling task, the teacher-scaffolding, and the student collaboration.

The modelling task used is specifically designed based on modelling principles that are aligned with reformed classroom practices (Lesh at el., 2003). The modelling task would provide the context for argumentation and collaboration, and the teacher functions as a facilitative coach to provide scaffolding at certain junctures of the modelling process. Student-student interaction in relation to the task as well as the teacher-student interaction in relation to the task would generate discussion around problem interpretation, variables, and strategies towards solving the problem. The students’ cognitive processing is manifested through the discourse as mathematical modelling behaviours of mathematical interpretations found within modelling stages of Description, Manipulation, Prediction, and Optimization. An ideal modelling pathway would have been the transition from Description to Optimization in a linear fashion based on much focused modelling efforts within each stage but the complexity of the task and the human dynamics involved would in reality make the process iterative. In this study, Description refers to attempts at understanding the problem to simplifying it which includes behaviours of drawing inferences from text, diagrams, formulas or whatever given data to make sense of the task details. It also entails students making assumptions from personal knowledge to simplify the problem as fitting the contextual parameters. Manipulation refers to behaviours of establishing relationships between variables, mathematical concepts, and task details through constructing hypotheses, critically examining contextual information, retrieving or organizing information, mathematizing, or using strategies towards developing a mathematical model. Prediction refers to the scrutinizing of the models towards verifying that they fit the parameters given and justifying them as workable models. Thus, unlike Figure 1, Verification is subsumed under Prediction. A new stage proposed is Optimization which refers to making improvements to or extending the models to reflect the enhanced solution that students produce through considering cost and material savings as a form of maximizing value.
5. Method

The main study employed a mixed-method design where both quantitative and qualitative data were collected and analyzed. The study was designed over two phases where Primary 6 students were involved in five different types of modelling tasks. This paper will trace the students' model development endeavours in one of the modelling tasks known as the Floor-Covering Problem. Each session lasted close to an hour.

Participants and Setting

The study took place in a neighbourhood school where two classes of Primary 6 students (total n = 80) and their respective mathematics teachers were involved. Prior to the study, both teachers and students had no experience in the PBL approach or had engaged in such mathematical modelling tasks during their normal mathematics classroom periods. The mathematics classroom instruction had mainly been the teacher-expository type.

The students were grouped into small groups comprising four or five students based on friendship and academic results. In each class, two groups had been identified by their respective mathematics teachers to be the target groups for the video-recording based on the teachers' perception that they were the more vocal students. The non-target groups were involved in doing the same things as the target groups except that they were not video-recorded.

A week before the commencement of the study, the teachers received a short training stint conducted by the researcher on facilitating students' learning based on the use of prompts for eliciting, supporting and extending students' thinking that were adapted from Fraivillig's (2001) article on Strategies for Advancing Children's Mathematical Thinking in the light of the modelling tasks that they would be facilitating.

Procedures

The modelling activities were carried out on almost a weekly basis in the Math Hub or Music Studio as it afforded space for group work. Students had access to stationery such as markers, rulers, huge writing papers, scissors and calculators as required by the tasks. Students were also provided with a KWD template (What we Know; What we Want to know; What we need to Do) as metacognitive aids to help them with their discussion.

The teachers acted as cognitive coaches and offered scaffolding when necessary to keep the cognitive engagement high or when the situation deemed it necessary for intervention. The students surrendered all the materials to the researcher at the end of the hour and a debrief was carried out with the students about the tasks.

6. Data Collection and Analysis

The four target groups were video-recorded for their group interaction in each session. An audio-recorder was also used to back up what could have been missed in the video-recording. However, video and audio recording was only carried out from the third session onwards. The first two sessions were designed as preparatory sessions for both students and teachers to be acquainted with the new pedagogical environment and be familiar with the presence of the researcher and the video-equipment. The other corpus of data included the students’ written work, the KWD templates and group reflections to triangulate with the analysis made from the transcribed data whenever possible.

For data analysis, an interpretive framework based on the protocol analysis method was employed to parse protocols (Schoenfeld, 1992; Goos & Galbraith, 1996) into broader modelling stages and finer episodes based on dominant problem-solving behaviours since its focus was to capture the process of solving a problem and identify the sequence of states the students progress through (Chi, 1997). Moreover, a task analysis of the modelling task had been done much earlier during the design of the problem task which served as a guide in matching the students' problem-solving actions towards interpreting their conceptualizations and mathematical meanings. A problem-solving coding scheme
was developed to code the problem-solving behaviours. Since the aim of this paper was to show the students' mathematical modelling by way of their conceptualized representations and mathematical relations, the coding scheme being not relevant with the aim of this paper is not attached.

The Modelling Task

The modelling task entitled the Floor-Covering Problem (see Figure 2) was a modification and expansion of a similar modelling problem by Gravemeijer, Pligge, and Clarke (1996).

The Floor-Covering Problem
You have been asked by your mother to suggest a covering for the floor of your study room. The room is rectangular and measures 4.3m by 3m. There are three ways to cover the floor. You can use the mat, carpet, or tiles but they are of different costs. Explain clearly and mathematically your best choice and how you arrive at your decision. Drawing diagrams may make your explanation clearer. (Note that pictures are not printed to scale)

In the Floor-Covering Problem, the context was one where the students had to determine an appropriate choice of a floor-covering material and its respective layout design in covering the floor of a study room. The dimensions of the room were given and there was a choice of materials, namely, carpet, mat and tiles that the students could use. The materials came with different dimensions and costs and there were also loose materials alongside their respective costs to be considered. An actual roll of serviettes was also made available in case the students need to use it to simulate the situation of unrolling the mat or carpet.

The main requirement in accomplishing the task lies in the statement "Explain clearly and mathematically your best choice and how you arrive at your decision". That implied the need for students to unpack what the meaning of "best choice" would be. There were two main expectations that were pre-determined as related to the students' development of models: One, the construction of layout designs that required a manipulation of dimensions of the floor-covering materials to fit the dimensions of the study room, and two, the ability to associate measurement and cost aspects. There were opportunities for students to factor personal knowledge or experiences and make assumptions as well. A plausible solution model would be one where the students would be able to attain a layout design that would fit the floor dimensions and at a reasonable cost. An optimized solution would be one where further improvements could be made towards cost and material savings.
Findings

Of the four video-recorded sessions of the four groups, it was unfortunate that the recording for one of the mixed-ability groups did not turn out successful due to a technical problem with the tape. As such, the findings of only three groups were presented. The three groups are labeled here as HA1 (Higher-Ability Group 1; n = 4), HA2 (Higher-Ability Group 2; n = 5) and MA1 (Mixed-Ability Group 1; n = 4).

Overview of Groups' Modelling Process

figure 3. Overview of students' modelling process

Figure 3 shows an overview of the groups' modelling process depicted as a timeline diagram across the various modelling stages as a simple interpretive framework. The bracketed numbers printed in certain stages imply the number of models the students had developed within those stages and the "T" implies the presence of teacher in providing scaffolding.
The timeline representation revealed three areas worth noting about the students' mathematical modelling process.

1. **The modelling process is iterative.** The pathways suggest that the modelling endeavours were not linear as all the groups had iterated to a previous stage at one time or another while some groups had iterated more than once. HA1 and HA2 had fewer iterations than MA1 (MA1 returned to the Description stage another three times).

2. **Focused inquiry determines modelling progression.** It was observed that groups that spend more time within a particular modelling stage delved deeper into the task. They elicited more details and asked more questions about the task and surfaced more alternative designs as well. Figure 3 also marks a distinction between the higher-ability groups and the mixed-ability group in that the former were more focused in their modelling endeavour as seen from the number of models developed within the Manipulation stage. The excerpts below show an example of the distinction between the HA1 and MA1 in keeping the inquiry focused during the first Description stage. HA1 contemplated about the meaning of "best choice" to gain a better understanding about their goals:

   55 S2 Why don't we calculate the best way and the cheapest way for each one first?

   56 S4 Can I say something? They never state to find the cheapest way, so we can choose not to.

   57 S3 Yes, "explain clearly mathematically your best choice" (referring S4 to the task sheet).

   58 S4 They say the best choice, not necessarily the cheapest.

   59 S3 What do you mean by the best choice?

   60 S2 The cheapest choice lah. The one that you spend the least money.

   The contemplation saw them defining what they wanted to achieve through giving meaning to the words "best choice" which kept them focused on their goal.

   On the other hand, MA1 encountered problems from the beginning as coincidentally all the four members were not able to conceptualize the unrolling of the carpet and mat material. They only saw that one of the dimensions for the carpet and mat were given and could not figure out what to do when the other dimensions were not printed.

   58 S2 4m by how much?

   59 S4 Let's go back to the thing (takes hold of the task sheet).

   60 S1 Let's put the materials we choose (referring to writing the materials down)

   61 S2 This is 4m by (…)? They never write 4m by what, so how we find the area? (holding on to the task sheet)

   62 S4 Maybe we don't count this one first? (as in skip that part first)

   ... ... ...

   65 S2 So carpet is 4 metres by what?

   66 S4 So that's the problem.

   67 S3 That is something we don't know. We don't know the carpet is 4 metres by how many metres.

   As seen in protocol lines (PL) 58, 61, 65, and 67, the students were asking one another about the "missing" dimensions but no one seemed to know what to do. Because they only knew about the dimensions of the square tiles then, they thus entered the Manipulation stage to work on the tiles first.

3. **Teacher scaffolding affects shift in thinking.** It was observed that teacher-scaffolding play a part in shifting the thinking of the students into certain stages. For example in MA1's case, the teacher in
trying to help the students make sense of the task details when the students were in the Manipulation stage shifted the discussion back into the Description stage. As well, the teacher in trying to get the students to talk about the comparison of costs and designs moved the discourse towards the Prediction stage. For HA1, the teacher scaffolded the students to think about refining their model that resulted in an enhanced model and that brought them into the Optimization stage.

Students' Model Development

The model development of the three groups is described as an expansion of the timeline diagram towards exemplifying the groups' conceptual and mathematical representations. Only the models conceived during the Manipulation, Prediction and Optimization stages are exemplified as consistent with the aim of this paper since these were the stages where the development of conceptual representations were more explicit.

<table>
<thead>
<tr>
<th>Modelling Stages</th>
<th>Model development (Developing conceptual representations)</th>
<th>Interpretation of models through students' reasoning and computations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manipulation</td>
<td>(a) Conceptualized laying out the carpet breadthwise and</td>
<td>(a) and (b). Reasoned about number of loose pieces needed to</td>
</tr>
<tr>
<td>Stage</td>
<td>patching the floor gap with three loose pieces as shown.</td>
<td>patch the gap: (a) is more cost efficient than (b) because only 3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>loose pieces were needed instead of 6.</td>
</tr>
<tr>
<td></td>
<td>(b) Conceptualized laying out the carpet breadthwise and</td>
<td>In asked out total cost of carpeting as: (4 x 3 x $12) + (3 x 30)</td>
</tr>
<tr>
<td></td>
<td>patching the floor gap with six loose pieces as shown.</td>
<td>= $152.</td>
</tr>
</tbody>
</table>
|                  |                                                         | (c) "Wow you 4m and pasted it this way. Here 4m. So you have to cut out  
|                  |                                                         | 4m..."                                                           |
|                  | (d) Conceptualized laying out two rolls of mat            | (d). It will be 2.5 times 3 plus another one (another roll), that will be 6 times 2 |
|                  | breadthwise.                                            | 2.5 x 3 x 2                                                     |
|                  | (e) Conceptualized laying out of mat lengthwise.         | (e). Dimensions of gap worked out as (4m x 0.5m) and (0.3m x 2.5m) |
|                  |                                                         | By aligning 1m edge of loose pieces to the gap, 7 loose pieces were |
|                  |                                                         | identified. Total cost of matting worked out as (4 x 2.5 x $11) + |
|                  |                                                         | (7 x $5) = $145.                                                |
|                  |                                                         | However, did not unroll until 4.3m.                             |
|                  | (f) Conceptualized laying out of tiles.                  | (f). Reasoned that 6 tiles in a row for 6 rows would be sufficient to |
|                  |                                                         | cover the floor. Worked out the cost of tiling as 9 x 6 x  |
|                  |                                                         | $3 = $54.                                                     |
|                  |                                                         | Cost of cutting tile not considered.                            |

Figure 4. Model development of HA1 during the Manipulation Stage
The Case for HA1. Figure 4 shows HA1’s model development during the Manipulation stage. Six models in terms of design layouts were developed each showing a different way to cover the floor. Models (a) and (b) show how the students conceived the carpet through unrolling it breadthwise that resulted in two ways to patch the 0.3m by 3m floor-gap depending on the orientation of the loose materials used. Interestingly, the students argued that model (a) was more cost efficient since three pieces of loose materials were needed instead of model (b) which required six pieces (“Because here 1 metre, then there’ll be lots of wastage. You’ll need more than 3”). This augured well with their goal which was to save costs. Mathematically, it was obvious that they were able to manipulate the dimensions of the floor-covering materials to fit the floor area and as well match the loose materials in patching the floor gap. In relating the cost to the area of carpet used, they multiplied the unit cost of the carpet by the area of the carpet and added the additional cost for the loose materials used to get the total cost of carpeting which amounted to $162.

HA1 then conceived another way to cover the floor by unrolling the carpet lengthwise as seen in model (c). They did not realize that this conceptualization had infringed the task parameters but nonetheless the idea was dropped because they realized that it had cost more. What model (c) had revealed more significantly was that the students did not perceive that the carpet could have been rolled all the way to 4.3m but instead they stopped short at 4m and then decided to use the loose materials to patch the gap they had created. Actually, this model was queried by a member and the reply from another member was “4.3, but they don’t sell in 0.3. They only sell per metre square” inferring that the students had taken “per metre square” to literally mean in full metres without considering the fractional parts.

In conceptualizing model (d) based on the mat material, the students used two rolls of mat and unrolled them breadthwise across the 3m of the floor. Since this layout design was costly, they dropped this option. For model (e), the students unrolled the mat lengthwise against the floor and stopped at 4m instead of 4.3m for reasons already mentioned. They worked out the dimensions of gap as (4m x 0.5m) and (0.3m x 2.5m). By aligning the 1-metre edge of the loose pieces to the gaps, 7 loose pieces were identified. The cost of matting was found to be (4 x 2.5 x $11) + (7 x $5) = $145.

The next conceptualization was the tiling of the floor seen as model (f). The students worked out the number of tiles needed per row and per column and found the cost to be more expensive as their working showed 9 x 6 x $3 = $162. There was no need to factor the service charge for cutting the tiles as this layout design was dropped.

The students reviewed and affirmed that their choice of floor-covering to be the mat based on evidence of design and cost (see excerpt below) in the Prediction stage:

224  S3  OK, so for the carpet we have $162, for the mat we have $145, and for the tiles, we have $162 again. So it is mat, correct? Mat, Method 2.

225  S2  Although for the loose pieces, it has lots of leftover, it is still the cheapest method.

234  S3  Thus our conclusion is that if we use mat, method 2, is the cheapest way although we have some pieces of mat left over.

The students then moved into the Optimization stage when they improved on their model as assisted by the teacher’s promptings:

261  T  So if here is 0.5 metres, here will be…

262  S3  0.2,

263  T  OK, so here is 0.2, right? So you'll have these pieces. Can you use these remaining pieces?

264  S2  You mean you can cut along the loose pieces?

265  T  It’s a carpet. What did they say? You can "further cut". (T points to the task sheet) Think about it.

266  S3  We can cut it as small as possible.
Further cut. Think about it.

(Pause as students try to figure out as they look at the task sheet)

OK, lets go back to this one. Carpet. 4 times 3. So how do we do it?

(S2 begins to unroll the serviette again model the situation)

If she is saying that we can use the remaining loose carpet to fix instead of buying another loose carpet to fix, that means instead of buying another loose carpet, we can use the remaining to fix.

The improvement with respect to cost and material savings was based on the following conceptualization in Figure 5, model (g), where instead of using three loose materials to patch the gap for models (a) and (e), the students determined that two loose pieces were sufficient as they could reuse the potentially waster materials. The final decision was still based on the mat material with a cost of $140.

The Case for HA2. HA2's model development endeavour during the Manipulation Stages is shown in Figures 6 and 7. They began with using the tiles first by conceptualizing as an 8 by 6 array (model (a)) before it evolved into a 9 by 6 array (model (b)) because of the need to fill the floor gap. The students reasoned that model (b) required the need to cut away the protruding tiles and asserted from personal knowledge that the cost of cutting the tiles to be $0.80 per piece. The cost of tiling was achieved by finding the cost of the 48 tiles and adding the cost of cutting the 6 pieces of tiles.

HA2's conceptualizations on using the carpet (model (c)) and the mat (model (d)) interestingly revealed what they perceive "per square metres" as. Both models (c) and (d) were designed as square layouts; 4m by 4m for model (c) and 2.5m by 2.5m for model (d). The reason for these designs was most probably due to a misconception of the term "square metres" to imply a square; a member argued "You want square metres. But everything is a square". Models (c) and (d) were not cost effective designs as they needed more loose pieces to patch the bigger gap as seen in model (e) (see Figure 7) which evolved from model (d). The extensive patching made them realize that that the square shaped designs were inferior models and eventually they conceptualized models (f) and (g) as revisions to the mat and carpet layout designs.

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<tbody>
<tr>
<td>Manipulation Stage 1</td>
<td>(a) 8 by 6 array</td>
<td>&quot;We'll need about 48 tiles to cover a portion of it, then there will be 0.3m left&quot;</td>
</tr>
<tr>
<td></td>
<td>(b) 9 by 5 array</td>
<td>&quot;So I want to cut it into 0.3 by 0.5, so that I can fit them here. So now I need 6 more. Multiply by 6.&quot;</td>
</tr>
<tr>
<td></td>
<td>(c) 4m by 4m square layout</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(d) 2.5m by 2.5m square layout</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(e) 9 by 6 array</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(f) 10 by 5 array</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(g) 10 by 5 array</td>
<td></td>
</tr>
</tbody>
</table>
In verifying their models towards making a decision for the choice of material, layout design and cost, the students did a pair-wise comparison in terms of cost and concluded that the mat material was the cheapest in the Prediction Stage. Selected protocols depicting the pair-wise comparisons are used to exemplify the situation:

170 S2 Compare square tiles and carpet? Which one is more expensive?
171 S3 So that means carpet is also out of the question. Mat and tiles, tile is nicer but then mat is cheaper.
172 S4 So which one is the best?
173 S3 To cover the same area, mat actually requires less money.
174 S4 But mat doesn't look as good.
178 S3 So between mat and tile which one will you choose? Mat is cheaper but tile is nicer.
182 S3 For tiles, you are forgetting that it is more expensive by $5.00. (*estimating from $5.55*)
186 S3 I think I'll go with tiles. Everybody agrees?

The comparison did not just centre on cost but also the aesthetic aspect as seen in PL 171 and 174. Although the group found that the mat material to be the cheapest (PL 173), they employed their
personal knowledge to argue that the difference between the cost of matting and tiling was small (see PL 182). They decided to choose the tile layout design instead citing that the aesthetic design was nicer and that using tiles would have long term benefits in terms of cost savings. This is evidenced by their conclusion in their writing "Tiles! It has a beautiful design, and even though mats come at a cheaper price, it does not have a nice design as compared to the tiles. Also, mats have to be washed regularly, whereas tiles have to be cleaned once in a blue moon. Overall, tiles will serve us better in the long run".

HA2, however, did not reach the Optimization Stage as they did not make a projection to mathematically show how much cost savings there would be as a result of switching to using tiles.

The Case for MA1. Figure 8 shows MA1’s model development endeavours during the various Manipulation Stages. The students were faced with the difficulty of visualizing how to cover the floor with the carpet material and model (a) was quite under-developed as it showed them able only to find the area of the floor and how to merge two loose materials to make 1m$^2$.

Faced with the difficulty, the only logical way to carry on was for the students to work on the tiles layout design in Manipulation Stage (2) since they knew the dimensions of the tiles (see model (b)). It was not until later that they clarified with the teacher about the unrolling of the carpet and mat that they began to conceptualize layout designs using the carpet and mat materials in Manipulation Stages (3) and (4) (see models (c) and (d) in Figure 11). Model (d) left a huge gap with the need to use many loose pieces to patch which should incur higher costs but because of calculation inaccuracies, the students found model (d) to be cheapest.

MA1’s workable model was arrived at in the Prediction stage based on the comparison between the three floor-covering layout designs and they chose model (d) as it was the cheapest. MA1 did not enter the Optimization stage because they did not make improvements through conceiving alternative designs that were more cost efficient.

<table>
<thead>
<tr>
<th>Modelling Stages</th>
<th>Model development (Developing conceptual representations)</th>
<th>Interpretation of models through students’ reasoning and computations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manipulation Stage 1</td>
<td>(a) Worked out the area of the floor to be 12.9m$^2$ through 4.3m x 3m. Reasoned that areas of two loose pieces would make 1m$^2$</td>
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</tr>
<tr>
<td>Manipulation Stage 2</td>
<td>(b) Reasoned that 8 tiles in a row for 6 rows would be sufficient to cover the floor. Worked out number of tiles to be 48. Did not work out the cost yet.</td>
<td>(b) Reasoned that 8 tiles in a row for 6 rows would be sufficient to cover the floor. Worked out number of tiles to be 48. Did not work out the cost yet.</td>
</tr>
<tr>
<td>Manipulation Stage 3</td>
<td>(c) Reasoned that the cost of carpeting was 4 x 3 x $12 = $144 Instead of using carpet loose materials, they used the loose mat material that worked out to be 3 x $5 = $15. In total, the cost of the mixed design was $159.</td>
<td>(c) Reasoned that the cost of carpeting was 4 x 3 x $12 = $144 Instead of using carpet loose materials, they used the loose mat material that worked out to be 3 x $5 = $15. In total, the cost of the mixed design was $159.</td>
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</table>
The aim of this paper was to trace the model development of students in the modelling process. Situating mathematical modelling in a problem-based learning instructional setting had enabled intense student-task interaction and student-student interaction to make visible the ways they develop models during the modelling process.

Insights on the Modelling Process

Each group’s modelling process was found to be iterative between the modelling stages but the number of iteration differed between groups. The iterative modelling process was consistent with other studies that showed the need to revisit problem information, test and revise approaches towards improving the models (Blomhøj, 2004; Doerr & English, 2003). The testing and revising of models also took place within modelling stages as evident from the evolution of the models and as some researchers assert as a modelling process being composed of sub-processes (Simon, Tzur, Heinz, & Kinzel, 2004). The modelling task in this research was seen as problem-driven as it had engaged intense student-task interaction and as well student-student interaction. As the students underwent the modelling iterations, it also made visible the meaningfulness of problem-solving in that it went beyond using the givens and mapping them towards getting a single correct solution but that the students were thinking more deeply about making their solutions workable and better (Doerr & English, 2003). The cyclic process that resulted in the refinement of models saw the production of at least a workable model verified in the Prediction stage for this particular modelling task. Not every group managed to enter the Optimization stage to attain an enhanced model suggesting that students tended to believe the workable model they had achieved sufficed. This could be a situation where students might not have thought hard enough or they thought they had done enough.

A fundamental aspect of the mathematical modelling process according to Llinares and Roig (2006) was to be able to recognize the underlying structure of the situation. It implies the recognition of the quantities and variables that were involved in the situation and how the students should manipulate these quantities and variables between them in achieving their goals. In the examples of HA1 and HA2, their ability to relate floor and material dimensions enabled a good visualization of working towards how to cover the floor. Being able to relate the various dimensions kept them focused in drawing up different ways to conceptualize layout designs. For MA1, however, their initial inability to perceive the dimensions impeded their progress. The students’ uncertainty and the teacher’s intervention to help them resulted in the group having more iterations during the modelling process. Notwithstanding, it does not mean that the mixed-ability students were disadvantaged. When students go through iterative processes or cycles, it would have meant that their conceptualizations have been put to the test repeatedly which in a sense is part of a developmental process in nurturing thinking. Moving back to the Description stage also implies that students could have met with unease as they evaluated their conceptualizations and thus returned to unpack more information as in the case of HA2. Reaching the Prediction Stage is testament of being able to get a workable model, a view consistent

7. Discussion

Figure 8. Model development of MA1 during Manipulation Stages 1 to 4

(d) Conceptualized laying out of mat breadthwise

(d) Unroll mat breadthwise

4.3m

0.3m

2.5m

1.5m

Reasoned that 12 loose pieces would be needed. Worked out the cost of matting for main area to be $2.5 \times 3 \times $11 = $82.50.

Worked out the cost of the loose pieces incorrectly. They took the area of each piece 0.5m² to multiply by 12 and then multiplied by $5 to get $60. Obtained $112.50 in total.
with Lesh et al (2003) that even weaker (academic) learners were able to achieve as their studies found them to be able to invent constructs and ways of thinking in modelling situations.

**Insights on the Models Developed**

Llinares and Roig (2006) pointed out that when the different quantities involved in the situation and the relationships between them were recognized, students would find a mathematical content or procedure which models the revealed structure. The more pronounced emerging models reported in this study showed how the students use a consistent structure: Area of floor x cost per unit area of material. This structure is viewed as a summation of two parts: (Area of floor covering material for certain amount of floor area x cost per unit area of material) + (Area of floor gap x cost of amount of loose material). The conceptual representations coupled with the mathematical relations bears similarities to Gravemeijer's (1997) notion of mathematical modelling as a form of organizing and translating where models emerge through the organizing and the related mathematical procedures as translation. In fact, the models featured in Figures 4, 7, 8 and 11 all show these two aspects being part of the structure mentioned.

A basic model that was developed became the tool to use regardless of the orientation of the material as laid out on the floor or the type of material used. In other words, once the structure was realized, it could be used in any similar situation that required the consideration for covering a floor space of a room. However, getting to recognize the structure allows for a workable model to be developed but not necessarily an enhanced model, after all as in this modelling task the floor area is fixed. What varies therefore is the conceptualization of the layout designs which accounts for the different costs. This would demand the exercise of representational fluency towards reusing materials to achieve the goal of maximizing value. Some scaffolding provided by the teacher could be helpful or a longer period given for students to persevere in the problem-solving could pave the way to improve the model. As in HA1's case, a little scaffolding questions provided by the teacher enabled them to transform their quantities into ways that helped them maximize value.

**Insights on Mathematical and Personal Knowledge**

Students are expected to draw on some of their curricular mathematics knowledge and apply to the modelling situations. The lack of certain aspects of their curricular knowledge could cause the students to reach an impasse or conceptualize inferior models as seen in this study. At the research front it is deemed a good thing because it surfaced what the students do not know or have misconceived. For example, in the case of the HA1 and HA2, both groups showed a lack of understanding of the unit of measurement "square metre" which resulted in developing inferior models initially. In another instance, MA1 in working out the area of the tiles was confused why the calculator showed 0.25 when they multiplied 0.5 by 0.5 as they thought multiplication should make the product bigger. At the social-learning level, it is also deemed a positive outcome because such a learning platform enables new learning to take place through students resolving such issues via their collaboration and discourse or through the teacher's scaffolding. In this light, the value of reform-oriented learning is this seen when flawed reasoning is surfaced and debated and through the problem-solving process enables students to edge closer towards what they believe are better solution models (Chan, 2009). These reasons concur with related modelling studies that found students developing important mathematical ideas and processes that they normally would not experience in traditional school curriculum (English & Watters, 2004; Zawojewski, Lesh, & English, 2003).

The personal knowledge of the students was also surfaced during the modelling process. For example, in HA2's case, they assigned a service charge of 80 cents for the cutting of each tile to fit the floor-gap when no such information was in the task sheet. In the Prediction stage, the pupils made decisions to choose the tiles as they deem the tiles to be more lasting even though they had found out that the mat was the cheaper means to cover the floor. Pace (2000) pointed out the importance of providing real-world situations for students to experience before they can come up with models related to the same context. The opportunity for students to factor their personal knowledge also makes them more engaged and interested in solving the problem. In a sense, the students' exercise of their personal
knowledge showed their trying to fit what was familiar to them into the problem-solving situation and developing highly context-specific solution strategies (Grevemeijer & Doorman, 1999).

8. Conclusion and Implications

The findings and discussion all point to a pedagogy that is vastly different from traditional problem solving. The use of a PBL platform serves important functions for mathematical modelling. It enables the interaction of its three important tenets; students, task, and teacher. The interaction implies the generation of a discourse rich in mathematical and metacognitive thinking that sees students getting involved in developing models through making conceptual and mathematical representations.

Students embarking on mathematical modelling go through different modelling stages. This shows the iterative nature of the modelling process. The iterative aspect enables them to evaluate and revise their models towards goal resolution. Going into a particular modelling stage provides a glimpse of the different modelling process students are involved in, from understanding and unpacking the task details, conceptualizing the models through manipulating the data, to verifying and making improvements to the models, suggesting the simulation of actual problem solving in the real world. As knowledge does not operate in a vacuum, students also bring what they know into the setting thus making the modelling process more meaningful and purposeful as in solving a real-life problem.

The models the students developed reveal the ways they organize the quantities and variables as relationships through their discourse in interpreting, analyzing, explaining, hypothesizing, conjecturing, comparing, and justifying. Emerging models become the tools for generating more and better models when students recognize the structure involved.

While the positives on mathematical modelling have been highlighted, to embrace mathematical modelling in a problem-based learning setting continues to be quite a challenge as the underlying tension between innovative approaches and traditional assessment modes has been and still is the case in issue for most countries. It will take a concerted effort by researchers, educators, and other stakeholders like parents and students to make the implementation a possibility and eventually a culture in the curriculum. More research is needed in this area to link theory and practice, to find an appropriate balance between contemporary and traditional approaches, and as well to train teachers as facilitators. These are but a few of the many action points to be considered. In writing this paper, it is a small but significant step in promoting mathematical modelling in the mathematics classroom.

References


