Teacher’s ways of thinking about students’ mathematical learning when they implement problem solving activities

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Abstract
We describe an experience with Mexican in-service teachers. In this experience the teachers designed didactic proposals to develop students’ mathematical knowledge. This proposal was used to teach the same course several times. After each implementation the teachers discussed the experience. We observed the cycles of refinement when the teachers implemented the proposals and discussed the results. In this process the teachers’ conceptions about learning and teaching mathematics emerged. The conceptions changed when the teachers shared and discussed the results amongst themselves. We used the Models and Modelling perspective to analyse instructional proposals. The documents produced by the teachers were the base to identify the evolution of ways of thinking about the students’ learning process. The teachers described in this paper are studying in a master’s program in mathematics education.

Key words: Modelling, In-service teachers, Cycles of understanding.

1. Introduction

Learning mathematics involves the development of mathematical knowledge and thinking. The students have to learn concepts, to develop abilities, skills, and habits of the discipline. In addition, they have to acquire abilities to transfer the knowledge and skills to new tasks and learning contexts. How can this be done in the classroom? What kind of mathematics teachers are required? What kind of educational programs, projects, and processes should be developed? How should mathematics teachers be supported to incorporate activities into their teaching practices that motivate students to commit to learning? When we discuss the need to improve a teaching and learning process there are several questions that arise and relate to all levels of participants—from students, to teachers, to developers, to researchers- of the education system. In this paper we discuss questions including: What kind of mathematical activities and learning processes do teachers emphasise in their mathematics instructional proposals when they have to include problems? How do teachers think students learn mathematics? How do teachers conceive their role and the role of their students during instruction? The results presented in this paper were derived from a Mexican teachers’ training program. This program includes Models and Modelling and Problem Solving perspectives as strategies to teach mathematics.

2. Literature Review

The Models and Modelling perspective (MM) is our framework. It reveals the complexity of the teaching of mathematics, learning, and problem solving (Lesh, 2010). MM conceives the learning of mathematics as a process of development of conceptual systems or models that are continually changing during the interaction between the individual and the problem or situation. The learning process involves a series of cycles of understanding where the conceptual systems are changing to more refined models. The early interpretation of the
situation usually changes because subjects gradually notice more relevant information. In these cycles the subjects think in different ways about givens, goals, and the possible path from givens to goals. The initial assumptions are incorrect or excessively limiting. The early cycles are barren and distorted compared with the later interpretations. According to Lesh and Doerr (2003), models are: “Conceptual systems (consisting of elements, relations, operations and rules governing interactions) that are expressed using external notation systems” (p. 10). For example, the students develop models or conceptual systems to describe or explain the behavior of a real situation or problem posed in the classroom. The students can also develop models to create a new problem or situation.

In the MM perspective the subjects develop models to understand the environment and situations where they are involved. According to this perspective, the process of externalization of the conceptions is a resource to understand the subjects' way of thinking. The models give information about the subjects’ conceptions of the situation. They show the early ways of thinking and how these evolve (Lesh, 2010).

Doerr and Lesh (2003) propose the use of model eliciting activities (MEAs) for subjects to reveal their thinking, prove, revise, and refine these ways of thinking with a particular purpose. MEAs are defined as real situations that involve conceptual tools. The subjects have to create models to solve them instead of giving short answers. The open situations, where the answer is not unique, are considered pertinent to develop knowledge. The subjects need to construct models to describe and explain the situations, the subjects do not produce “the answer”. MEAs are similar to many real life situations. According to Lesh (2003),

Model Eliciting activities differ from traditional textbook word problems. This difference results from the fact that, beyond the computational skills that most traditional textbook word problems are intended to emphasize, it is often the case that the main thing that is problematic for students is to make meaning of symbolically described situations (p. 3)

Traditional problem solving becomes a special case of model-eliciting activities. The textbook problems contain symbolic descriptions of situations, data, relations, unknowns, etc. In Model Eliciting Activities the subjects have to identify the relationships and to use different representations in order to communicate them to other persons.

The subjects of the Models and Modelling perspective research can be students, teachers, developers, researchers (or instructors) themselves or other education decision makers. In this sense, the teachers should create models to understand the students’ learning and teaching processes. Doerr and Lesh (2003) propose that we have to create contexts in which teachers can express and modify their ways of thinking about the teaching and learning processes. “The essence of the development of teachers’ knowledge, therefore, is in the creation and continued refinement of sophisticated models or ways of interpreting the situations of teaching, learning and problem solving” (p. 126). MM perspective gives information about how teachers interpret their practice, how their ways of thinking evolve over time and how these interpretations influence the teacher’s practice.

Lesh (2003) proposes that teachers have to interpret situations in the context of their actual practice. The teachers need to simultaneously address mathematical content, pedagogical strategies, and psychological aspects of a teaching and learning situation; for example, those aspects related to the conceptual learning in the individual and social settings. They have to share ideas about teaching and learning with colleagues, to seek their replication and reuse ways of thinking in multiple contexts. They need to be able to judge for themselves whether their interpretations and consequent actions are moving towards desired ends in particular contexts.

3. Methodology
Six Mexican in-service teachers with experience at the high school level took part in this process. However, the data presented here were taken from two teachers (Kent and Jim). They were studying a master’s degree program in Mathematics Education. The teachers had already discussed Problem Solving Perspective (Schoenfeld, 1994) among other psychological theories to understand the learning and teaching process. They had discussed the advantages of encouraging problem solving in the classroom to promote mathematical knowledge and thinking. They could differentiate between the routine and non-routine problems (Santos, 1997). They discussed the role of teachers and students in this perspective. They also had taken part in problem solving experiences, as students of the master’s program.

The teachers were asked to design an instructional proposal through real situations or problems that would foster the students’ development in mathematical knowledge and thinking. The teachers had to implement the proposals in the classroom. Several questions were raised: What mathematical knowledge and thinking should students exhibit at the end of a mathematics course? What type of activities or problems should I select to support the development of students’ knowledge and thinking? In preparing the proposal the teachers had to consider the context of their institutions, which involved reviewing the objectives, plans, curricula, and mathematics books. The proposals were used in courses taught in different cycles by the teachers. The proposals were implemented, modified, refined one or two times.

We were interested in the process of development of the proposals created by the teachers because we were studying the teachers’ models or ways of thinking about the students’ learning process. In this sense we took all the documents produced by the teachers. Models and Modelling perspective was used to analyse the teachers’ development in the processing of didactic proposals. The analysed products were the didactic proposals and the discussions derived from them. First, we detected two different ways of thinking about the students’ learning process. We also identified cycles of understanding of the proposals which were continually changing during the interaction among the teachers, the peers, the instructor, the classroom and the proposals.

The communication was very important. The teachers had the opportunity to share the proposals with their instructor. They discussed together the design of the proposals, objectives, issues, pedagogical strategies chosen and psychological aspects of a teaching and learning situation as those mentioned by Lesh (2003). The teachers also received support from peers in group sessions. We observed the teachers’ work during one year. The teachers were designing, modifying, implementing and refining the proposals. There were two implementations in the classrooms.

4. Results and Discussion of Results

The didactic proposals were analysed and the role of the included problems. We identified two different types of proposals. The first type included problems as final activities of the didactic proposals. The goal was to use them like an instrument to assess the mathematical learning. We referred to them as Problem as Assessment. The second type included problems as a vehicle for promoting the development of mathematical knowledge and thinking. We referred to them as Problem as Media Learning.

Didactic proposals: Problem as Assessment

There were three proposals characterized by this type. These proposals included mathematical definitions, properties, examples, and exercises. The role of the teacher was to present and expose some examples to illustrate the use of the mathematical concepts and the properties. The teachers encouraged the students to repeat similar activities as we can see in Fig. 1 (taken from one of the instructional proposals). These activities can be considered exercises or routine problems because they were proposed after the teacher explained how to
solve similar problems. The context of these activities was the mathematics, i.e. the questions were about mathematical subjects.

The teachers justified the inclusion of exercises in the didactic proposal. They thought that the students had to build mathematical knowledge before solving problems. They also argued the need to include concepts before including procedures.

Activity 1

The teacher, in plenary, explains to the students how to manipulate and use the following equations

\[(x-h)^2 = 4p(y-k) \quad (y-k)^2 = 4p(x-h)\]

The goal is that students relate the equation with the graph.

Fig. 1 Example of the activities included in the first type of the proposals. This one was elaborated by one teacher who will be called Kent.

The teachers taught the students the prerequisite ideas and skills in mathematical contexts. They taught general content and then they showed how to use preceding ideas, skills and heuristics to solve routine mathematical problems. In these proposals the teachers’ role in the classroom was to initiate and answer questions from students and to guide the activities towards some objectives.

The problems included in the proposal by the teachers were the so-called routine mathematical problems (Fig. 1). The non-routine problems were included at the end of the didactic proposal. The role of them was focused on the transfer of knowledge to new situations. The teachers used these last problems to assess the learning of mathematics. The problems were similar to those from textbooks (Fig. 2). The teachers hoped the students could manipulate algebraic expressions and use them to solve problems in different contexts (Problem 1 is an example).

Problem 1. The trajectory described by a projectile launched horizontally from a point situated \(y\) meters above the ground with a speed \(v\) m/s is a parabola whose equation is

\[y = -\frac{gx^2}{2v^2}\]

Where \(x\) is the horizontal distance from the launch site, and the value of \(g\) is approximately 9.81\text{m/s}^2. The origin is taken as the base of the projectile.

Under these conditions a stone is thrown horizontally from a point 3m above the ground with an initial speed of 50m/s. Calculate the horizontal distance to the starting point and draw the trajectory of the projectile.

Fig. 2 This is a non-routine problem included in Kent’s instructional proposal. It is considered a non-routine problem because the teacher did not show the students how to solve it. The problem was a challenge for the students.

The limited achievements of students in solving non-routine problems allowed Kent to analyse their didactic proposal and the instructional activities. The discussion between the teacher and the instructor permitted the teacher to identify two processes that were not connected in the proposal: the learning of knowledge and the assessment. In other words, the teacher had not connected the resolution of exercises related to the equation of the parabola and its graph, and the process of solving non-routine problems associated with parabolic motion. The teacher discovered that the assessment process of learning had become a process of mathematical learning. Solving the non-routine problems became for the students another stage of development of mathematical content understanding.

After discussion with the instructors, the teacher recognised that the problem solving process should be used to generate learning instead of being used only to assess the acquired
knowledge. This way of thinking allowed him to redefine the role of problems in the proposal, the objectives and the concept of assessment and transference of knowledge. He modified the model or way of thinking about the process of learning and he reorganised the instructional activities before implementing again.

**Didactic proposals: Problem as Media Learning**

There were three proposals characterized by this type. One proposal of this type was designed by Jim. The proposal was based on solving non-routine problems involving situations that were familiar to the students. The designed non-routine problems involved the analysis of functional relations between quantities in different contexts such as trips, shopping, and phone calls. The problems were introduced from the beginning of the proposal until the end of it. The teacher explained that the aim was to develop mathematical understanding through problem solving.

In this didactic proposal, Jim justified the need for students to be able to integrate different representations and conceptions to understand mathematical concepts. Fig. 3 is an example of a second version of a designed worksheet to promote the learning of the concept of function. The activity includes the following, that students have to examine a table, look for patterns, and write an equation. The activity does not include exercises related to algebraic manipulation.

The teacher implemented the proposal with students who had taken previous traditional algebra courses. The students worked individually to answer the questions proposed in the activity. Jim’s role was focused on reviewing the answers given by students; he validated students’ solutions, and answered questions. In some sessions Jim proposed to manipulate algebraic expressions as homework, but he observed that students did not engage in this activity. The algebraic expressions proposed were not related with the principal activity.

Jim discussed the results obtained by the students with the instructor. This discussion allowed Jim to evaluate and refine the didactic proposal. He redefined objectives and pedagogic strategy to be pursued later. He reviewed the exercises included in the student’s homework in order to connect them with the main activity in the worksheet. He decided to incorporate work in pairs and group discussions in the classroom. Jim constructed a second version (Fig. 3).
Jim implemented his second version of the didactic proposal with another class. He organised students in pairs to support each other in the learning process. Problem solving and group discussions were aimed at guiding the session to his objectives for the lesson. His role as a teacher changed and was no longer focused only on the work of individual students. However, when the teacher communicated the results of the implementation, he realised that his role had focused again on validating procedures and answering questions, rather than questioning the knowledge of students, even when they were doing the expected activity.

Jim found that working in pairs could help the students to assess and refine their answers to the questions and understand the general situation in the activity. These settings helped to communicate individual ideas, generate arguments, and assess ideas.

In these two types of didactic proposals (Problem as assessment and Problem as Media of Learning) we can observe some differences and similarities. In the first version of the proposals, the teachers included routine and non-routine problems, but the order of the inclusion was different and without connection between them.

The two types of proposals exhibited three important aspects: the role of the students, the role of the teachers and the mathematical knowledge. In the first version of the two types of proposals the role of the students was passive, the role of the teachers was active and the mathematics knowledge was reduced to learn definitions of mathematical concepts or algorithms. The teachers related these aspects in ways that were not useful to reach the goals in terms of the mathematical knowledge and thinking that they wanted the students to develop. The teachers expected the students could transfer the developed knowledge to different situations without any previous experience. There were similar conceptions about learning mathematics, the role of the students was passive, and the role of the teacher was 

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**As a team complete the following table. Calculate the price to pay for \( x \) number of kilograms of tortillas, make a graph, and answer the questions. Consider that a kilogram of tortillas cost 12 pesos.**

<table>
<thead>
<tr>
<th>Kg of tortillas</th>
<th>Amount to pay</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{4} )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3 ( \frac{1}{2} )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>7.5</td>
<td></td>
</tr>
<tr>
<td>8 ( \frac{3}{4} )</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
</tr>
<tr>
<td>25.5</td>
<td></td>
</tr>
</tbody>
</table>

a) If a store produces 215.5 kg of tortillas a day. How much money is made if everything is sold? 

b) Express the equation that allows you to calculate the cost of any number of tortillas. 

c) Use this expression to calculate the cost of 51 kg of tortillas. Explain the procedure. 

d) The following equation \( \frac{2}{3}y + \frac{3}{4}x \) is used to calculate the total cost of tortillas. The "\( x \)" represents the amount in kg of tortillas, "\( y \)" is the total amount due and the constant value is the cost of the paper to wrap the tortillas. What is the expression to calculate the total amount due "\( y \)"?

e) How many kg of tortillas can be bought with 780 pesos? 

f) How many kg of tortillas can be bought with 20 pesos? 

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**Fig. 3 Second version of the activity designed by Jim**
protagonist. These proposals exhibited the teachers’ conception about the way that the students must learn mathematics. This way included that the students did only one kind of activity, for example, the repetition of algorithms and procedures. The teachers did not consider the importance of developing abilities and habits of thought in the classroom such as: generating conjectures, argumentation, and the meta-cognition process.

In the second version of the proposals the teachers showed a new cycle of refinement about their understanding of the students’ learning process. They considered the importance of developing abilities and habits of thought in the classroom. The teachers recognized the role of the student in the process of their knowledge and thinking development; therefore, they encouraged a more active role for students in the learning process. They recognised the role of the interactions among the students to develop knowledge, and they encouraged the work in pairs and group discussion. The teachers identified the need to relate the conceptual knowledge and the ability to use it in other situations in the classroom; so, they reorganised the activities and problems in the didactic proposal looking for students to transfer their experience (Doerr and Lesh, 2003).

When the teachers were involved in the design and implementation of the didactic proposals they exhibited their conceptions about the mathematics and the mathematical learning process. These conceptions were changing while they were discussing the results. According to Models and Modeling Perspective (Lesh, 2010), we require this kind of situation for teachers to reveal, prove, revise and refine their thinking, and therefore to modify and extend their conception of the mathematics and the mathematical learning process. The teachers need to interpret or assess the students’ thinking, plan appropriate sequences of instructional activities, select and modify or extend tasks in the light of student comprehension (Doerr and Lesh, ibid).

5. Conclusions

The questions raised in this paper are: What kind of mathematical activities and learning processes do teachers emphasise in the mathematics instructional proposals when they have to include word problems? How do teachers think students learn mathematics? How do teachers conceive their role and the role of their students during instruction? These are closely related to the type of didactic proposals designed and implemented by the teachers.

The opportunity to share the proposal with colleagues and instructors, allowed the teachers to evaluate, revise and refine their thinking about teaching and learning mathematics. The teachers had the opportunity to problematize the didactic proposal and instructors had the opportunity to know their beliefs about mathematical knowledge, the assessment and learning that mathematics teachers had.

Regarding the third question posed in this paper we found that the teachers did not problematize the activity of the students in the classroom to promote learning. The conception was reduced to answer the questions and validate the procedures. The opportunity to discuss the experience with colleagues and instructors permitted teachers to realise the need to change the roles attributed to the students and the teacher. The teachers were analyzing the assessment of the student learning too. Teachers began to understand that assessment was not simply to notice that “six students correctly answered and four did it incorrectly”. They are learning about how to create collaborative environments to support the learning process. Instructors are also in the process of learning, creating models that allow an understanding of the evolution and development of the teacher learning experience. Instructors are sharing ideas in seminars, which serves to generate models which support the development of teachers.

We agree with the Models and Modelling perspective that is concerned with the complexity of situations that teachers have to consider to be better teachers and about how
they have to create models to describe, explain, manipulate, predict, and control their own teaching practice. Learning requires the modification of mental models that are created and modified when the subject can analyze and communicate them to others. Collaborative environments can support this process.

References


