Developing Mathematical Competences, Learning Linear Equations, Functions and the relation among these Concepts

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Abstract
In this article we report part of the findings of a research project focused on the developing of mathematical competences and knowledge about concepts as linear equations, functions, variables, unknown quantities and solution. The research topic is algebraic thinking. We discuss about the research questions: What are the cycles of understanding exhibited by students when they solve an algebraic problem? What kind of competences do the students exhibit in each cycle? How can we develop competences? The participants were college students from University of Quintana Roo, Mexico. They were 18 and 19 years old and they were starting degree courses in social science. They solved problems in a Problem Solving environment encouraged in the classroom. The students could extend, revise and refine their own conceptual systems. The Problem formed part of a series of problems implemented in the classroom. We used qualitative methodology to analyze the results obtained. We used Models and Modelling Perspective to analyze student's procedures and learning processes. The results show us cycles of understanding exhibited by the students when they solved the Problem. In each cycle we observed how college students were developing understanding of the mathematical concepts while they were solving problems. We concluded that the meanings can evolve to a more structured and organized way while students solve problems, and a Problem Solving environment is useful to develop Mathematical competence.

Keywords: Mathematical competences, linear equations, functions.

1 Introduction
In this era of rapid advances both in science and in technology, there is a demand to have people not only more informed, but also better educated. We require competent citizens able to make contributions for the further development of the society. Various questions emerge upon this theme: What does it really mean to have competent citizens? Is mathematical knowledge important to be competent? What does it mean to develop mathematical competence? How should we develop mathematical competence in the classroom?

Some researches (De Corte, 2007; Kilpatrick, 2002) consider that students need to develop mathematical competence. Learning mathematics should not be reduced to memorize concepts, procedures and algorithms. The National Council of Teachers of Mathematics (NCTM, 2000/2003) considers that mathematical knowledge, abilities, habits and skills are important in competent citizens. The social environment, the role of the professor and the tasks are considered essential to get them (NCTM 2000/2003).

Learning mathematics is not a linear process. The Models and Modeling perspectives MMP (Lesh, 2010) conceives the learning of mathematics as a process of development of conceptual systems that are continually changing. The learning process involves a series of cycles of understanding where the students’ conceptual systems are changing from poorly structured models to more refined models. When the students solve problems or situations, they have to analyze information, use different modes of representation, develop and justify conjectures. The students have to express, test, and revise their ways of thinking, which are utilized to explain, describe, predict, and calculate. In this article we report part of the findings of a research project focused on the developing of mathematical competences and knowledge about concepts as linear equations, functions, variables, unknown quantities and solution. The questions we try to answer in this article are: What are the
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cycles of understanding exhibited by the students when they solve an algebraic problem? What kind of competences do the students exhibit in each cycle? How can we develop competences? We use Models and Modeling contributions (Lesh, 2010; Lesh & Doerr, 2003) to analyze student's performance. The procedures we describe in this article were used to solve one of the problems designed to develop mathematical competences and knowledge.

2 Review of Literature

Obtaining mathematical competence implies acquiring a mathematical disposition characterized by five aspects (De Corte, 2007, p. 20, 21): an accessible base of knowledge, well-organized and flexible, the use of heuristic methods and meta-knowledge, positive mathematics-related beliefs, and self regulatory skills. The students have to understand mathematical concepts, operations and relationships. They have to think in a flexible, accurate, efficient and appropriate ways. They need abilities to represent, formulate and solve mathematical problems. The students have to think, reflect, explain and justify logically.

The inclination to see mathematics as a sensible, useful and valuable discipline is important as well as the students can feel confidence in knowledge and skills (Kilpatrick, 2002). Mathematical competence implies that the students have the capacity to transfer these skills and knowledge to new tasks. The acquisition of mathematical competence according to De Corte (2007) requires an active-constructive, self-regulated, situated, and collaborative process of learning.

The National Council of Teachers of Mathematics (NCTM, 2000/2003) provides arguments similar as those raised above. Mathematics is considered important for all citizens. NCTM (2000/2003) proposes rethinking the design and development of curriculum, instruction and assessment of learning processes for all educational levels. It suggests the creation of a problem solving environment in the classrooms where the students should have the opportunity to learn mathematics as a living, dynamic and constantly evolving discipline. Researches as De Corte (2007), Kilpatrick (2002), Schoenfeld (1985) and Santos (1997) do not reduce the learning mathematics to memorize some procedures; they agree that learning mathematics means to develop concepts, skills, abilities and habits. De Corte (2007) and Kilpatrick (2002) call all these aspects: mathematical competence.

Problem Solving perspective (Schoenfeld, 1985) proposes the creation of a community of learning where the students construct a deep understanding of mathematical ideas and processes by engaging them in doing mathematics. The role of the professor is fundamental in this community to encourage students to develop significant answers, justifications and explanations. The design and use of the problems in the classroom is important to promote the learning of mathematics concepts, to develop strategies, procedures, control abilities, beliefs and heuristics.

Learning mathematics is not a linear process; the understanding of mathematical ideas does not develop one at a time (Lesh & Yoon, 2004). Concepts, skills, abilities and habits are related components of the mathematics learning. They are developed by students when they solve situations. The interaction with problems and situations involves to analyze and to interpret information, to identify particular cases, to use different modes of representation, to develop and justify conjectures (Lesh, 2010). The Models and Modeling perspectives (MMP) conceives the learning of mathematics as a process of development of conceptual systems (Lesh & Doerr 2003) that are continually changing during the interaction between the individual and the problem or situation. The conceptual systems are used to describe, explain, communicate, and to predict the behavior of other systems. The learning process involves a series of cycles of understanding. In the early stages of the development, the conceptual systems are characterized as diffuse, poorly differentiated, and poorly coordinated (Lesh, 2010). In these cycles, the conceptual systems or models are changing to more refined models. Conceptual development is a gradual and contextualized process.

The activities and the individual work, in teams and in groups, allow students to represent interpret, speculate, evaluate, communicate, build, explain situations to other people. During the interaction with problems or situations, the students pose questions and seek strategies to find the answer. They share with other classmates the ideas about the strategies implemented and about the outcomes. They make explicit the criteria used to evaluate and compare those strategies. Students go through multiple stages of understanding the situation and its solution; each phase is a refinement of
the previous one. Everyone communicate their ideas and conceptualizations of the situation and actions for the solution. The thinking of the individual students is strongly influenced by the learning communities. The students’ development typically occurs through participation in communities of practice (Lesh & Yoon, 2004). The product of the activities allows the teacher to have important elements to describe characteristic features of the students’ thinking about these situations and about the mathematical concepts used.

3 Methodology

The participants in this research were 20 students enrolled in the first semester of university courses. The students were studying social science careers. They were 18 and 19 years old. The mathematical knowledge background of the students was from high school. The high school mathematics programs from Mexico include the learning of algebra language, equations, system of linear equations (SEL), function, variable, unknown, and methods to solve simultaneous linear equations. For example, the students have to learn the methods of elimination, substitution, and graphing to solve linear of system equations, up to three unknowns. They have to learn how to elaborate tables, graphs, and symbolic expressions.

The Problem described in this article forms part of a series of problems designed in a research project focused on the development of mathematical competences (De Corte, 2007; Kilpatrick, 2002); it involves concepts as function, variable, unknown, solution and linear equations. The Problem was implemented in a Problem Solving (Schoenfeld, 1985) environment where students worked in small teams to solve it. The problem is a situation that incorporates several questions. Each question and the context are a problem, like text problems (Lesh, 2010). The questions were elaborated to guide the students’ understanding about a contextualized situation.

The class session took two hours. During this time, the students began reading the Problem in teams, each one containing three participants. Advances were discussed at various times by the presentation of the teams to their classmates and the class session ended with a global analysis of the Problem. Finally, the homework assigned was to write an individual report demonstrating how the solution of the problem was getting. One of the reports is included in this article (figures 2-4) as an example of the process followed to solve the Problem. 15 students did similar reports, in terms of the discussion showed in this article.

The main focus we are interested to show here is about problem solving process rather than problem solvers alone, because we observed that all the teams exhibited a similar process to solve the Problem. The Problem is the following.

Problem. The cost of a barrel of Maya Petroleum is $395.00, and the cost of a barrel of Brent Petroleum is $545.00.

a) If you mix one barrel of each brand of petroleum, how much does a barrel of this resulting mixture cost?

b) If you mix two barrels of Maya Petroleum, and one barrel of Brent Petroleum, how much does one barrel of this resulting mixture cost?

c) If you mix one barrel of Maya Petroleum and two barrels of Brent Petroleum, how much does one barrel of this resulting mixture cost?

d) If you mix 15 barrels of Maya Petroleum, and 20 barrels of Brent Petroleum, how much does one barrel of this resulting mixture cost?

e) How much should you mix from each brand of petroleum if you want the mixed barrel that you produce to cost $450.00?

f) How much should you mix from each brand of petroleum if you want the mixed barrel that you produce to cost $500.00?

g) How much should you mix from each brand of petroleum if you want the mixed barrel that you produce to cost $600.00?

The questions lettered a-d can be solved in an arithmetic form. The aim is to offer the students the opportunity to observe the relationships among the data and identify a pattern in the structure of the operations. The questions lettered e, f, and g require the structure of the following linear equation:
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\[
\frac{395x + 545y}{x + y} = b
\]  

\(\text{equation (1)}\)

In the above equation, \(x\) corresponds to the number of barrels of Maya Petroleum, and \(y\) corresponds to the number of barrels of Brent Petroleum. The letter \(b\) corresponds to the cost of one barrel of the mixed petroleum.

The methodology used to analyze the data was qualitative. Data were collected from professor’s notes and the student’s reports. We used Lesh (2010) and Lesh and Doerr (2003) perspective to analyze the process of solving the Problem. In particular, we recognized the cycles of understanding. We used some De Corte (2007) categories to identify the types of competences.

4 Results

4.1 Process to solve the Problem

4.1.1 First cycle of understanding. Arithmetical procedures (an accessible base of knowledge): looking for patterns

The teams read the Problem. They did not have difficulties to answer the four questions \(a, b, c\) and \(d\) because of their arithmetic nature. However the procedures were not organized in a systematic way. Some of the teams were using calculator, they did not register the operations. For example, they added 395 to 545 and divided the result by two in order to answer question \(a\) of the Problem.

The teams used trial and error procedures to answer the question \(e\) (Problem). They chose possible quantities of barrels as solution, and carried out the corresponding operations. For example: they chose 10 and 5, calculated 395*10+545*5 and wrote the result (6675) on the notebook. After this, they calculated 10+5, they wrote the result (15) on the notebook, and finally they made the division: 6675/15. They compared the result obtained with 450 and they began to look for quantities to get closer to 450 (expected result). They repeated this procedure several times. Because they could not get 450 when they operated with many different elected quantities, they asked the professor if the question \(e\) had an exact solution.

The students demonstrated understanding of the relationships among the data in the Problem, they noticed some dependency among all the quantities. They recognized the structure of the relationships included in the text of the Problem and the unknown quantities. Nevertheless, they did not consider writing the arithmetical procedure in algebraic language to solve the question \(e\) (Problem). They did not identify the functional relation, the procedure was arithmetic one. However, it is important to underline that this procedure showed that the teams had an accessible base of arithmetical knowledge (De Corte, 2007).

The professor asked the teams to organize the operations to solve questions \(a, b, c\) and \(d\). The organization of the operations would help them to systematize the information. The role of the professor in the classroom was important in this stage to solve question \(e\). The questions that the professor asked the teams were: Which operations did you realize to answer the questions \(a, b, c\) and \(d\)? How can you write the whole arithmetical procedure? Are there quantities that did not change in every calculation? Do you identify quantities that are changing? The teams wrote the procedures in different ways (Figure 1) and these procedures were the focus of a group discussion.

![Figure 1: Procedures to solve questions a-d of the Problem.](image)
The team found the results 470, 445, 495 and 480.71.
Some questions as “Is there any other method that you can use to find the solution?” “Can you write the procedures using equations?”, and one procedure showed to the classroom by one team to organize arithmetic procedures (similar to the individual report in Figure 2) helped the group to utilize algebraic language, and write the equation $\frac{395x + 545y}{x + y} = 450$ (equation 1).

![Figure 2: Individual report. Procedure to find the solution to $a-d$ in the Problem. This procedure is part of a final report, so we can see explanations.](image)

The challenge became to find the solution of the equation. The teams had only one equation with two variables so they were confused about how to find the solution. They asked the professor: “Is this a correct equation?” “Is there another equation?” “How can we know if we have all the equations we need to solve the problem?” “How can we find the solution?” The teams did not have an accessible base of algebraic knowledge to solve the question $e$ of the Problem.

4.1.2 Second cycle of understanding. Algebraic procedures: writing equations and using different mathematical representations to solve questions e-g, Problem. Understanding the solution

The professor intervened again in the classroom to promote discussion, and asked the students some questions as follows: How can we find the solution or solutions for this equation? Can we suppose that $x$ has a specific value? Can we substitute this [specific] value in the equation? Can we find the value of $y$? What happens if I take several values for $x$? What happens if I make a graph? Can we draw a graph? Can we find the solutions in a graph? How many solutions does the question $e$ have? The students were encouraged to use different methods or representations to solve question $e$ (Problem). The students waited to find only one solution. The professor generated a group discussion in order to find a procedure to solve the equation, to find some possible solutions and to do the graph of the linear equation (similar to the individual report in Figure 3).

Because the teams always were looking for only one solution, they asked the professor: What kind of equation is that? Why can I give values to $x$? How can I choose the values? It was difficult for the students to understand and explain the relation among the graph, the equation and the concept of infinity quantity of solutions. The group discussed about this relation during the session.
It was easier for the class to find the solution to question f (Problem). The previous activity was useful to understand this question. The teams found that there were an infinite number of solutions again, and they obtained some of them from the equation (similar to the individual report in Figure 4).

When solving question g, the teams expressed surprise and asked: What is the significance of a straight line? (The students did a graph similar to the Figure 5) Can we talk about negative quantities for barrels of petroleum? Is there a solution to the problem? These questions provided an advantage to discuss in the classroom. Some other questions arose as: Which is the smallest quantity we can choose as price for a barrel of petroleum instead of 450? When can the problem have a solution? The students used graphs again to analyze the question g, Problem.
4.1.3 Third cycle of understanding. Writing procedures to communicate and argument the procedures and solutions

The students had to write an individual report as homework (Figures 2-5 are examples of this reports). The students did not explain all the results obtained in the classroom. They did not write all the conclusions. They wrote their own understanding about the procedure to solve the Problem. All the students’ individual reports had these characteristics. For example, in Figure 3 we can see the explanation about the procedure to solve question e of the Problem: “To solve this problem, you have to use the following formula. Later, you only have to find the value of x and y to get the results of how many barrels of Maya Petroleum and how many barrels of Brent Petroleum you will use for the mixture. Later, I can add these two quantities to get the total quantity of barrels”. The student (Figure 3) was trying to explain the procedure and the “formula” used, but the student did not write something about the solution of the question e of the Problem. The answer “this problem has an infinity quantity of solutions given by this formula” was not registered in the reports, although it was the conclusion in the classroom. The students only circled the answer (Figure 3). The students did not differentiate between equation and function, they named formula to both.

In the reports some students explained that they followed similar procedures, they only changed quantities. For example, in the question f (Figure 4) they used the same previous equation. The students registered the formula (function); they did some substitutions and the graph. They used two solutions (points of the plane) to draw the graph. However, they did not write anything about the infinity quantity of solutions.

In Figure 5 we can observe the procedure to solve it. The student in this report explained that in order to get the answer, he only needed to change the quantity of 600 in the equation (2). Then, the student said about the solution: “Since it gave me a negative answer, the Problem does not have a solution.”

5 Discussion of results and Conclusions

The questions that underline this article are: What are the cycles of understanding exhibited by the students when they solve an algebraic problem? What kind of competences do the students exhibit in each cycle? How can we develop competences? These questions can be discussed as following.

In the first cycle of understanding the procedures used to solve the Problem show that the students had arithmetical competence. They had an accessible base of arithmetic knowledge that permitted them to undertake the problem, identify information, and recognize patterns and relationships. In this case, they could answer questions a-d. However, the student’s arithmetical thinking was not useful to recognize equations immediately. The procedures to answer questions a-d needed to be reorganized.

In the second cycle of understanding, the role of the professor was important to promote understanding and to encourage discussion and reflection in order to access to the algebraic thinking. The type of interventions used by the professor helped the students to develop mathematical knowledge and mathematical competencies. For example, the questions asked by the teacher in the classroom helped students to reflect on the concept of solution and its relationship with other concepts, such as the unknown quantity and variable. They also, promoted to integrate the process of graphing in the procedure of solving the equation. This was useful in order to develop different heuristic methods to solve equations (graphs, tables, for example), and discuss the mathematical concepts in depth.

The role of the professor in the classroom was essential to reformulate, contextualize, discuss, and analyze the problem. The constantly monitoring of the procedures by the professor was important to identify the mathematical knowledge of the students, the cycle of understanding, and the formation of pertinent questions.

In the third cycle of understanding the students wrote and explained the procedures and solutions by themselves. They did not give deep argumentations about the solutions. However, their conceptual system was more refined in this third cycle than in the first one. The reports contained equations and procedures to solve them. The students could use different representations and they could find solutions. The reports that students did as homework (Figure 2-5) exhibit the difficulties and some progress about the understanding of concepts like function, linear equation, infinite number
of solutions, variables, and finding the solution to an equation. The role of the professor, the Problem, and the environment created in the classroom were important to help students to develop these mathematical competences and knowledge (Lesh, 2010; Lesh & Doerr, 2003).

The students had the opportunity to think in flexible, accurate and appropriate ways. They could represent, formulate, reflect, explain and justify logically. Regarding accessibility, well-organization, and flexibility, mathematical concepts such as variable, equation, solution of a linear equation and some others were studied in depth. Concerning heuristic methods, the use and integration of different forms of representation was promoted. The professor encouraged self-regulation skills, for example, the students had the opportunity to express their ideas through discussions, argumentations, and group presentations. The students were constantly reminded to validate their own problem solving procedures. They were also reminded to continuously go back and review the initial question stated in the word problem. The students could extend, revise and refine the conceptual systems.

The students need to have more experiences in solving similar problems that have infinity quantity of solutions. It would be useful to understand concepts as function, equation, variable, solution, unknown and the relationships among them. When there are a lot of experiences, the knowledge tends to be organized around the abstractions; in this case, it could be organized around algebraic thinking.

**References**


