Learning Mathematics Through Mathematical Modelling

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Abstract
In a mathematics education course at the University of Gothenburg, students were asked to develop modelling tasks or modelling situations for each others. There are many reasons for encouraging the development of peer tutoring among students. When explaining something to a class mate, students must clarify their own thinking in order to give an explanation and must be prepared to have misconceptions confronted and corrected through discussion and listening. In general students learn much more if engaged in the teaching of a course. But will the receiving students learn what was intended or maybe something quite different? In this article I will discuss how this activity was carried out by two groups of students (one teacher group and one student group) and what they thought they learned from it. I will also discuss how the possibility to use technology enables learning of mathematics in a new way.

Keywords: Peer tutoring, mathematical modelling, technology

1 Introduction

To teach matters of learning and teaching principles to prospective teachers is a huge task. There is so much content and information to cover in order to prepare prospective mathematics teacher for the upper secondary school level and usually rather few teaching hours in which we can teach it. In a mathematics education course for prospective mathematics teachers, we decided to try to combine several course objectives such as: Learning how to use a wide spectrum of technolgical resources, learning how to engage and challenge students’ mathematical thinking, and learning how to create learning situations which are relevant for most students in upper secondary schools.

Technological resources

We wanted them to learn different aspects of GeoGebra since it is such an excellent tool for teaching and learning many different branches of mathematics. It is free, portable (runs on any machine that runs Java), and combines Algebra, Geometry and spreadsheet number theory. We also wanted them to learn how to use digital cameras and free software for splitting a film into images. All students had access to cameras and/or cellular telephones which they could use to film with. We also introduced them to the software VirtualDub, which enables you to split avi files into images. All films are made of many images, from roughly 25 – 150 images per second. It is not surprising that there exist software that can separate the images from each other again.

Students

There were 24 students in this mathematics education course, they had all been studying mathematics at the university level for two semesters, and some of them (although not all) also had studied another teaching subject such as physics or chemistry. In our course, they have read and discussed article about theories on learning as well as papers on mathematical modelling (e.g. Sfard, 1991; Lingefjärd, 2006). We decided to split them up in six groups with four students in every group. Some of the students were known to be good in mathematics, some of them already knew about GeoGebra, some of the students knew about software for editing and handling movies, and so forth. We tried hard to make the groups as equally strong, with respect to these variables, as possible.
Three groups were chosen as teaching groups (presenting tasks) and three groups were chosen as students groups (receiving and solving tasks). One teacher group and one student group were paired.

Student activity

All three teaching groups developed interesting teaching materials. What I will present in this paper is just one task out of three and perhaps not the best or most accurate task. Nevertheless, it is an interesting activity in the sense that it combines most, if not all, of the course objectives mentioned above.

2. Theoretical framework

Studies on student learning (Ramsden & Entwistle, 1985; Johnson et al., 1989, Johnson & Johnson, 1991, Ross & Cousins, 1993, Marton & Booth, 1997) suggest that students adopt at different times and in different circumstances, different approaches to learning. In groups of two or more individuals, students work together, share and clarify ideas. Through talking to each other about subject matter, students can discover what they know and what they do not understand and ‘make sense’ of what they are learning. Some studies show that peer or co-operative learning forces students to engage in higher order thinking, which includes application, analysis, synthesis and evaluation (Ellis & Whalen, 1990). Of course there is no certain way to declare that peer or co-operative learning is the best way to learn. It seems that the more we learn about different ways of learning, the more complex and vague the picture becomes.

The reader might find that it has been somewhat bewildering, our bringing together findings on qualitative differences in the way in which learning is experienced that originate from studies of learning in widely differing educational contexts, that of infants, preschool children, secondary school pupils, and university students, and moreover from such widely differing cultural contexts as Britain, Sweden, China, Uruguay. Our assumption is that a phenomenon, such as learning as experienced, can be described in the terms of the complex of differing dimensions of variation identified. (Marton & Booth, p. 54, 1997)

In order to learn something certain conditions must be met. Learning situations are always experienced within a framing of different circumstances, such as a context, a certain time, a place, and so forth – while a phenomenon is experienced as abstracted from or transcending such an anchorage. But learning is also closely connected to the possibility of variation.

Variation theory makes it possible to analyse teaching and learning in commensurable terms, which implies that ‘what the teacher intends the students to learn’, ‘what is made possible to learn in a lesson’ and ‘what the students learn’ are connected and described in a similar way. From a variation theory position, learning is defined as a change in the way something is experienced, seen, or understood. A fundamental assumption is that the learner, in one way or another, experiences what is learned. The educational system aims at developing the learners’ capability to handle various situations, to solve different problems, and to act effectively according to one’s purposes and the conditions of the situation. However, the possibility of acting on, or handling, a situation depends upon how we make sense of it. We act in accordance with how we perceive the situation. This is affected by our previous experiences, but the experiences we see as relevant are also affected by how we experience the situation. ‘We try to achieve our aims, not in relation to the situation in an objective sense but in relation to how we see it’ (Marton et al., 2004, p. 5).

When working with prospective teachers, it is of interest to regard what they see as possible to learn from a situation both in the sense of their own learning, but also in their role as future teachers. In variation theory, learning is seen as becoming able to discern critical features of an object of learning. The object of learning is the definition of a competence or understanding of something, for example a particular content taught in a mathematics lesson. The object of learning is consequently not the same as learning objectives in a course: it is not the subject or the content taught and learned but rather the capability connected to that particular knowledge.
The second framework I like to use was proposed by Vinner and Tall (1981), which says that with each mathematical concept is associated a concept definition and a concept image. This framework is natural and quite easy to understand:

Many concepts which we use happily are not formally defined at all, we learn to recognise them by experience and usage in appropriate contexts. Later these concepts may be refined in their meaning and interpreted with increasing subtlety with or without the luxury of a precise definition. Usually in this process the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. But the total cognitive structure which colours the meaning of the concept is far greater than the evocation of a single symbol. It is more than any mental picture, be it pictorial, symbolic or otherwise. During the mental processes of recalling and manipulating a concept, many associated processes are brought into play, consciously and unconsciously affecting the meaning and usage. (pp. 151 – 152)

The concept definition can be the stipulated as a definition assigned to a given concept. Let us say that we like to define a circle with centre in \((a, b)\) and radius \(r\) as the algebraic definition \((x – a)^2 + (y – b)^2 = r^2\). The concept image of that circle, on the other hand, will be a nonverbal representation of any individuals understanding of the concept circle. It includes the “visual representations, the mental pictures, and the experiences associated with the concept name” (Vinner, 1999, p. 68). Most, if not all, humans probably have a rich experience of the concept circle which fits into the concept image.

According to Vinner to acquire a concept one has to form a concept image for it. Merely learning the definition of a concept does not guarantee that the concept itself will be acquired. Many mathematics instructors expect their students to develop a concept image based on a definition of the concept delivered in class or from a definition given in a textbook. This may not always be true. Vinner also refers to the concept image as an ‘evoked concept image’, that is, the mental image or a memory that is evoked in the individual’s mind when the word related to the concept is heard by the individual. The definition of the concept may have no or little role to play initially but later may become functional once the concept image is formed. For example, the word ‘triangle’ may evoke the concept image of a geometrical figure of a triangle in an individual rather the definition of a triangle ‘a figure with three sides and three vertices’.

An individual may require several inputs (other than a definition) to help form a concept image. For example to help an individual understand the idea of a triangle, the teacher may present several drawings or pictures of triangles, expecting the learner to ‘see’ some common features (invariant properties), among all the figures, namely, three sides, three vertices etcetera. The same idea may be explored in a dynamic geometry environment by drawing a triangle and dragging one of the vertices to create an animation, a continuous process, through which the learner will be able to see the invariant properties. Extending this a little further, we may want the learner to conjecture that the sum of the three angles of a triangle is 180°. Presenting several triangles on paper and making them measure the three angles could be one way. In a dynamic geometry environment, however, by dragging the vertices of a triangle the learner can ‘see’ in the algebra view, that the sum remains 180° irrespective of the shape of the triangle. In this study I hope to highlight the fact that exploring a concept in a dynamic geometry environment can play a vital role in shaping the student’s concept image. It is also my intention to show that the two frameworks, Concept Definition and Concept Image together with Variation Theory, are complementary.

Variation Theory can be seen as sprung from Phenomenography which is related to radical and social constructivism in that people perceive reality subjectively. Contrasted with the heterogeneity of constructivism, phenomenography is a homogenous perspective, and differs further by emphasizing the qualitative. Phenomenography's ontological assumptions are subjectivist in that the world exists and different people construe it in different ways; and with a non-dualist viewpoint. There is only one world that people experience in many different ways (Bowden, 2005; Marton & Booth, 1997). Phenomenography's research object has the character of knowledge; therefore the ontological assumptions also become the epistemological assumptions. We could call that experience an awareness of a phenomena or perhaps a concept image.

Sfard and Linchevski’s (1994) theory of reification and Sfard’s (1991) theory of process/object duality scaffolds Tall and Vinner’s theory of Concept definitions. The theory of
reification posits the existence of three stages of concept formation: (a) interiorization; (b) condensation; and (c) reification. The first two stages represent the operational aspect of mathematical notation and the last stage its structural aspect. According to Sfard (1991) the structural conception of a mathematical notation is static whereas the operational conception is dynamic and detailed. To understand the structural aspect of a mathematical concept is difficult for most people because a person must pass the ontological gap between the operational and structural stage. Sfard (1991, p. 3) distinguishes between the words “concept” and “conception”. According to Sfard the term concept represents the mathematical, formal side of the concept and conceptions the private side of the concept.

This seems to draw on Tall’s and Vinner’s (1981) theory of concept image and concept definition. They suggest that when we think of a concept something is evoked in our mind. Often these images do not necessarily relate to a concept definition even if the concept is well defined theoretically. The collection of conceptions is called the concept image. In Tall’s and Vinner’s theory, concept image is the whole cognitive structure that is associated with the concept. It is my view that this also could be called the unique view or awareness I have of a phenomena.

It is my conclusion that the theory of Variation, sprung from Phenomenography and the theory of concepts and concept images therefore are complementary.

Metacognition will be a natural output, when students are asked to reflect about their own thinking. There are several professional definitions of metacognition, which may give you different entries into how to think about your own thinking.

- Metacognition - is the process of planning, assessing and monitoring one’s own thinking; the pinnacle of mental functioning (Cotton, 2001).
- Metacognition - having (cognition) and having understanding control over, and appropriate use of that knowledge (Collins, 1994).
- Metacognition - is an awareness of oneself as “an actor in his environment, that is, a heightened sense of the ego as an active, deliberate storer and retriever of information. It is whatever “intelligent weaponry the individual has so far developed” is applied to mnemonic problems (Hacker, 2001).

3. Results e Discussion

Teacher group task

The students in this group went to a gym with a basketball hall and while one of them were practicing distance shots with a ball against the basket, the others in the group filmed the shots with different cameras and cellular telephones from several distances. When they were satisfied with the shooter’s performance and the content in the films, they went to a computer lab at the university and transferred all the films to a computer, watched all the short films and selected the best film in terms of quality. After that they downloaded the free software VirtualDub (http://www.virtualdub.org/) and used it to split the three second long film into frames. See figure 1 and figure 2 for two such frames. There are several more frames between these two.

![Image 1](http://www.example.com/image1.jpg)  ![Image 2](http://www.example.com/image2.jpg)
Researcher: So why did you choose this specific activity? What do you see as the possible object of learning? What will your peers understand or learn from this activity?

Student 1: Our first objective of learning is of course that our peer students should learn more about mathematical modelling. But we also think that they will learn about specifics in the modelling process, specifics such as that every point in the graph has a certain value which defines the point’s position in a Cartesian coordinate system. When you look at the different representations of the same object in GeoGebra, you understand more about that object.

Researcher: Please elaborate on that.

Student 1: I am referring to Sfard’s theory on dual nature of mathematical conceptions. It must be better to see different representations and be able to vary between them in order to understand more about them...

Researcher: Concept image...?

Student 2: Well, we think that when you look at that ball’s position, thereby imagining the trajectory of the ball, then you understand something about how the single images are part of a whole film. This is represented for thirteen positions of a ball. So you understand more about the connection between reality and the images and that connection is exactly the mathematical model. It connects to your concept images of several mathematical objects, I would say.

It is obvious that the teaching group chiefly was focused on the mathematical modelling activity as their object of learning. For them this whole complex process with many different specifics became reification into the mathematical model. The students were also interested in using more representative modes in order to develop concept images further. Whether these concept images were about the concept of mathematical modelling or about the concept of second degree functions is harder to say. Besides being software for teaching geometry and algebra, GeoGebra also contains several other possibilities suited for teaching different branches of mathematics. One such possibility is to insert an image anywhere in the coordinate system and then use it as a background. Another one is to mark a point at that image and then select “copy the coordinates to the input line”, e.g. A = (-2.02, -0.56). From the input line, you can easily relabeling the coordinates to A1 = (-2.02), B1 = (-0.56) and thereby make them part of the spreadsheet. Once in the spreadsheet, the set of points can be used to create a List of points, which then becomes an object. In our example, the ball’s centre act as the marking part in every picture and with every frame put in the same place, GeoGebra present us with a set of data points as in figure 3.

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<thead>
<tr>
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Figure 3: The motion of the ball represented in two ways
Student 3: And if you take any of the points (in GeoGebra) and drag it a little bit off, then you get another trajectory. See here, the fitted curve moves. So, by varying the position of just one single point, the mathematical model can indicate that the shooter made a goal or maybe missed to make a goal. It will teach you even more about mathematical modelling and how careful you must be when you measure or register data. (See figure 4).

\[
f(x) = -0.31x^2 + 0.38x + 1.48
\]

Correlation coefficient = 0.91

**Figure 4:** Observe that the trajectory of the fitted curve, the correlation coefficient and the formula for the polynomial will all change if one point is moved.

Researcher: What about the concept image...?

Student 4: This is clearly the concept image of a mathematical model which develops, perhaps together with the concept image of how single images are part of a whole film. I begin to believe that there is seldom a single concept definition in your mind and therefore also several concept images existing side by side. Nevertheless, I would argue that you, through this activity, do understand more about the connection between reality and the images and that connection is exactly the mathematical model. It connects to your concept images of several mathematical objects, I would say.

It seems as if the students hold an intuitive sense of the variation possibilities offered by the technology. It is also obvious that variation can be represented in many different ways. In the student statement above, it is the concept of position which is varied in different representational forms. That variation works on several concept images, according to Student 4. It was also notable that for the students in the teacher group the reality now was transferred into the film, with part of that reality expressed by the frames inside GeoGebra.

As indicated, the students in the corresponding student group were given 13 frames, illustrating the shot from the start and up to the peak of the ball’s trajectory towards the basket. They were asked to construct a mathematical model based on the ball’s position in the frames and then use the model to predict if the shot was a goal or not.

**Student group solving**

After some initial discussion in this group about the procedures and command structure of VirtualDub and GeoGebra with regard to this specific problem, they inserted the images into GeoGebra and then created the set of data points, and started to analyze the situation:

Student 1: Now when we have all the points in the spreadsheet and have created the List of point, we can use the regression command FitPoly[List1, 2]. If we do that, with a
polynomial of degree 2, we should get a curve fit that shows a hit or a fail. Do you follow?

Student 2: Well, I consider this to be an excellent way of introducing and using the concept of equations of second degree. This is actually the first time that I understand why we actually teach the students about them in the upper secondary school. Amazing!

Researcher: And what do you think that you are learning while you do this? What do you see as the object of learning?

Student 1: There are several things of course, I was not aware of the possibility to just go away and make a film and the split it into images and just say to the students: Analyse this situation. It is just so cool!

Researcher: So what did you learn? How?

Student 2: I also learned much more about second degree equations and I will definitely use part of this when I teach an introduction to second degree functions next time, it is awesome. We can all use this film to illustrate that!

Researcher: You know that we have talked about theories of learning connected to variation, concept definitions and concept images...

Student 3: oh, you meant that. Well, I guess that we are varying several aspects here and that we are building our concept images... But I need to think more about that before answering more deeply.

It is interesting that when students are divided into one “teacher group” and one “student group”, they also start to act as teachers and students. The student group were not at all interested in discussing learning theories and their connections to the activity they were engaged in. We also see that the student group actually assumed a different object of learning compared to the teacher group. The object of learning can obviously be seen from three different points of view: the “teacher’s”, the “student’s” and the researcher’s. The object of learning seen from the point of view of what the teacher or, in this case, the students who acted as teachers, is what is called the intended object of learning. Compare this to the “intended curriculum”. The intended object of learning was connected to mathematical modelling and the correlation between data points and regression curves.

Any teacher hopefully always has a particular goal and intention about what the students should learn, for instance that students are able to understand that a function must be continuous in order to have a derivative everywhere. On the other hand, what students (the learners) actually learn is the experienced (lived) object of learning and that can be observed during the learning situation by the students’ expressions, or after the lesson in an exam. Obviously, it is not necessary that the intended and the experienced (lived) object of learning coincide. In this activity, the students in the students group, experienced another object of learning, namely an example of how second degree equations can describe ball trajectory paths.

The object of learning seen and analysed from the researcher’s point of view, implying ‘what it is made possible to learn’, is the enacted object of learning. The enacted object of learning describes what features of the content are possible to experience during a “lesson”. The variation used to elicit the features, the values within a dimension of variation, is the range of change (Watson & Mason, 2006) and the way it is possible to discern the variation is through different patterns of variation.

For me, as a researcher, it was most obvious that the students now experienced a way of working with technology that can enable their own teaching in their future teacher professions. Some subset of their object of learning was of course related to mathematical modelling and the selection of suitable curves for the curve fitting process. For me that was in the details, while the overall picture was about new challenging working methods and opportunities when technology is at hand. Altogether, these results created an opportunity for a challenging seminar with all the prospective teachers where we discussed what they had learned and what the intentions were. That was probably the most vivid discussion I had ever seen in a seminar, especially when students from the teacher
group tried to “teach” their peers in the student group a clear distinction between variation, concept definitions and concept images.

4. Conclusion

It seems that this is an excellent example of how modern technology enables us to construct challenging mathematical activities directly from situations around us. It seems hard to imagine that this problem could have been constructed and presented by one group of prospective teachers to another group of prospective teachers without technology at hand.

Nevertheless, the experiment with students constructing problems for others enables someone to study the learning outcome and thereby observing the difference between the groups knowledge of the object of learning and between my own expectations of the learning outcome.

The more complex a teaching and learning situation gets the harder it is to analyse the situation in stereotypes or easy commensurable terms. Whenever we try to teach something, we should be aware of the fact that students or humans never perceive reality, since here is no reality outside our notion.

We cannot separate our understanding of the situation or our understanding of the phenomena that lend sense to the situation. Not only is the situation understood in terms of the phenomena involved, but we are aware of the phenomena for the point of view in the particular situation. Furthermore, not only is our experience of the situation molded by the phenomena as we experience them, but our experience of the phenomena is modified, transformed, and developed through the situations we experience them in. (Marton & Booth, p. 83) (Italics in original)

The most beneficial outcome from this teaching experiment was the conceptual growth among the students in the teacher group. Suddenly, they all became as doctoral students, with interest and curiosity focused on learning theories they saw in action in the activity they created. Clearly, they all understood what was meant by the fact that “variation affects the concept image of the object”. The students in the student group were much more trapped in the statement in the first line of the quotation above. One conclusion from this fact is that perhaps all students should act as “teachers” now and then, perhaps it is just in a metacognitive position we can understand the full implication of the learning activity we are dealing with.

References


